3 (Sem–1/CBCS) MAT HC 2

2022

MATHEMATICS

(Honours)

Paper : MAT-HC-1026

(Algebra)

Full Marks : 80

Time : Three hours

***The figures in the margin indicate
full marks for the questions.***

1. Answer **any ten** :   1×10=10

   (a) Find the polar representation of
   $z = -3i$.

   (b) State De Moivre's theorem.

   (c) Let $z_0 = r(\cos t^* + i \sin t^*)$ be a complex
   number with $r > 0$ and $t^* \in [0, 2\pi)$.
   Write down the formula for n distinct
   n^{th} roots of z_0.

Contd.

(d) Identify the quantifier, set of context and property in the statement, "Every student in this classroom is at least 5ft tall."

(e) Define implication. Give an example.

(f) Prove by contradiction "There is no greatest integer".

(g) Let A and B be two sets, write when $A \times B = \phi$. Justify your answer.

(h) What is domain and range for the function $f(x) = \tan x$.

(i) What are the options about the solutions of a system of linear equations?

(j) Determine h such that the matrix $\begin{bmatrix} 2 & 3 & h \\ 6 & 9 & 5 \end{bmatrix}$ is the augmented matrix of a consistent linear system.

(k) State True **or** False with justification : "Whenever a system has free variables the solution set is infinite."

(l) Write down the system of equations that is equivalent to the vector equation

$$x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(m) Define Pivot positions in a matrix.

(n) Prove $\vec{U} + \vec{V} = \vec{V} + \vec{U}$ for any $\vec{U}.\vec{V}$ in $\mathbb{R}^n$.

(o) Write the system of equation as a matrix equation

$$3x_1 + x_2 - 5x_3 = 9$$
$$x_2 + 4x_3 = 0$$

(p) Given, $A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Compute $x^T A^T$ and $A^T x^T$.

(q) A is an $n \times n$ matrix. Prove statement (i) $\Rightarrow$ statement (ii).

(i) A is an invertible matrix

(ii) $\exists a$ $n \times n$ matrix C s.t. $CA = I$

(r) A is an $n \times n$ matrix

Fill in the blank :

If two rows of A are interchanged to produce B, then det $B =$ _____ .

2. Answer **any five** : $2 \times 5 = 10$

(a) If $z_1 = 1 - i$ and $z_2 = \sqrt{3} + i$. Express $z_1 z_2$ in polar form.

(b) Write the 'converse' and 'contrapositive' of the following statement :

"For real numbers x and y, if xy is an irrational number then either x is irrational or y is irrational."

(c) Why may we use the contrapositive of a statement to prove the statement instead of direct proof? Justify using truth table.

(d) Produce counter examples to disapprove the following :

(i) For $x, y \in \mathbb{R}$, $|a| > |b|$ if $a > b$

(ii) For any $x \in \mathbb{R}$, $x^2 \geq x$

(e) Express the empty set as a subset of $\mathbb{R}$ in two different ways.

(f) Express $\mathbb{N}$ as the union of an infinite number of finite sets I_n indexed by $n \in \mathbb{N}$.

(g) Give an example of a relation that is not reflexive, not transitive but is symmetric.

(h) State True **or** False with justification :

An example of a linear combination of vectors $\vec{v}_1$ and $\vec{v}_2$ is $\frac{1}{2}\vec{v}_1$.

(i) Prove that the following vectors are linearly dependent

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

(j) Evaluate the determinant by using row reduction to Echelon form

$$\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$$

3. Answer **any four** : 5×4=20

(a) Compute $z = \left(1 + i\sqrt{3}\right)^n + \left(1 - i\sqrt{3}\right)^n$.

(b) Prove that the power set of a set with n elements has 2^n elements. Write down the power set of $S = \{a, b\}$.

(c) Prove that the equivalence classes of an equivalence relation on a set X induces a partition of X.

(d) Prove $(1 + x)^n \geq 1 + nx$ for $x \in \mathbb{R}$ such that $x > -1$ and for each $n \in N$. Give the name of this inequality.

(e) Balance the chemical equation using vector equation approach the following reaction between potassium permanganate $(KMnO_4)$ and manganese sulfate $(MnSO_4)$ in water produces manganese dioxide, potassium sulfate and sulfuric acid.

The unbalanced equation is

$KMnO_4 + MnSO_4 + H_2O \rightarrow MnO_2 + K_2SO_4 + H_2SO_4$
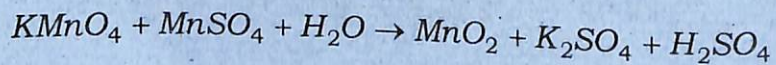
(f) Find the value of h for which the set of vectors is linearly dependent

$$\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$$

(g) Let A be an $m \times n$ matrix. Prove that the following statements are logically equivalent.

(i) For each $b \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.

(ii) Each $b \in \mathbb{R}^m$ is a linear combination of the columns of A.

(iii) The columns of A span $\mathbb{R}^m$.

(iv) A has a pivot position in every row.

(h) Use Cramer's rule to compute the solutions to the system

$$\begin{aligned} 2x_1 + x_2 \quad &= 7 \\ -3x_1 \quad + x_3 &= -8 \\ x_2 + 2x_3 &= -3 \end{aligned}$$

4. Answer **any four** : $10 \times 4 = 40$

(a) (i) Prove $\prod_{\substack{1 \le k \le n-1 \\ gcd(k, n)=1}} \sin \dfrac{k\pi}{n} = \dfrac{1}{2^{\phi(n)}}$

whenever n is not a power of a prime. 5

(ii) Solve the equation

$$z^7 - 2iz^4 - iz^3 - 2 = 0$$ 5

(b) For any three sets A, B and C, show that

(i) $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$ 5

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ 5

(c) Define graph of a function verify that the set $\left\{ (x, y) \in \mathbb{R}^2 : x = |y| \right\}$ is not the graph of any function. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax^2 + bx + c, a \ne 0$. Show that the function is neither one-one nor onto.

$2+2+6=10$

(d) Let $X = \mathbb{R}$ and let

$R = \left\{ (x, y) \in R^2 : xy = 0 \right\}$. When $x \in \mathbb{R}$ is related to $y \in \mathbb{R}$? Define reflexive, symmetric, antisymmetric and transitive relation with examples.

$2+2+2+2+2=10$

(e) If $A \subseteq N$, what is the least element of A? State and prove Division Algorithm.

$2+1+7=10$

(f) (i) Solve the system : 5

$$x_1 - 3x_2 + 4x_3 = -4$$
$$3x_1 - 7x_2 + 7x_3 = -8$$
$$-4x_1 + 6x_2 - x_3 = 7$$

(ii) Suppose the system 3

$$x_1 + 3x_2 = f$$
$$cx_1 + dx_2 = g$$

is consistent for all possible values of f and g, what can you say about the co-efficients c and d. Justify.

(iii) Suppose a 3×5 co-efficient matrix for a system has three pivot columns. Is the system consistent? Justify. 2

(g) (i) If $\vec{U} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\vec{V} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

Display $\vec{U}, \vec{V}, \vec{U} - \vec{V}$ using arrows on an xy graph. 3

(ii) List five vectors in the span $\{\vec{v}_1, \vec{v}_2\}$

$$\vec{v}_1 = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$ 2

(iii) Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Does $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span $\mathbb{R}^4$? Justify. 5

(h) (i) Describe all solutions of $A\vec{x} = \vec{0}$ in parametric vector form

$$A = \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ 5

(ii) Does $A\vec{x} = \vec{b}$ have at least one solution for every possible $\vec{b}$ if A is a 3×2 matrix with two pivot positions ? 2

(iii) Prove that if a set contains more vectors than the number of entries in each vector then the set is linearly dependent. 3

(i) (i) Define linear transformation. Give an example. 2

(ii) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then prove T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$. 3

(iii) Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which is a horizontal shear transformation that leaves e_1 unchanged and maps e_2 into $e_2 + 3e_1$. 3

(iv) Show that T is a linear transformation by finding a matrix that implements the mapping

$$T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$$ 2

Contd.

(j)   (i)   Find the inverse of the matrix A (if it exists) by performing suitable row operations on the augmented matrix $[A : I]$ where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}.$$

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(ii)   Find the volume of the parallelopiped with one vertex at the origin and adjacent vertices at $(1, 0, -2)$, $(1, 2, 4)$ and $(7, 1, 0)$.

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(iii)   Let the transformation

$T : \mathbb{R}^2 \to \mathbb{R}^2$ be determined by a 2×2 matrix A. Prove that if S is a parallelogram in $\mathbb{R}^2$ then

$\{\text{area of } T(S)\} = /\det A / \{\text{area of } S\}$

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