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3 (Sem–3 /CBCS) MAT HC 1

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–3016

(Theory of Real Functions)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any ten*** parts : $1\times10=10$

   (a) Is every point in I a limit point of $I \cap Q$?

   (b) Find $\lim\limits_{x\to1} \dfrac{x^2-x+1}{x+1}$.

   (c) Let $f(x)=sgn(x)$. Write the limits $\lim\limits_{x\to0^+} f(x)$ and $\lim\limits_{x\to0^-} f(x)$.

Contd.

(d) Let $p:\mathbb{R}\to\mathbb{R}$ be the polynomial function

$p(x):=a_n x^n + a_{n-1}x^{n-1}+\cdots+a_1 x + a_0$

if $a_n>0$, then $\lim\limits_{x\to\infty} p(x)=$?

(e) Let f be defined on $(0,\infty)$ to $\mathbb{R}$. Then the statement "$\lim\limits_{x\to\infty} f(x)=L$ if and only if $\lim\limits_{x\to 0^+} f\left(\frac{1}{x}\right)=1$" is true **or** false.

(f) Let $A\subseteq\mathbb{R}$ and let $f_1, f_2, \ldots, f_n$ be function on A to $\mathbb{R}$, and let c be a cluster point of A. If $\lim\limits_{x\to c} f_k(x)=L_k$, $k=1, 2, \ldots, n$,

then $\lim\limits_{x\to c}(f_1\cdot f_2\cdot \ldots\ldots \cdot f_n)=$?

(g) Is the function $f(x)=\dfrac{1}{x}$ continuous on $A=\{x\in\mathbb{R}: x>0\}$?

(h) Write the points of continuity of the function $f(x)=|x|$.

(i) "A rational function is continuous at every real number for which it is defined." Is it true *or* false ?

(j) "Let f, g be defined on $\mathbb{R}$ and let $c\in\mathbb{R}$. If $\lim\limits_{x\to c} f(x)=b$ and g is continuous at b, then $\lim\limits_{x\to c}(g\cdot f)(x)=g(b)$." Write whether this statement is correct or not.

(k) The functions $f(x)=x$ and $g(x)=\sin x$ are uniformly continuous on $\mathbb{R}$. Is fg uniformly continuous on $\mathbb{R}$? If not, give the reason.

(l) A continuous periodic function on R is bounded and ______ on $\mathbb{R}$. *(Fill in the blank)*

(m) "The derivative of an odd function is an even function." Write true **or** false.

(n) Write the derivative of the function $f(x)=|x|$ for $x\neq 0$.

(o) If f is differentiable on $[a, b]$ and g is a function defined on $[a, b]$ such that $g(x) = kx - f(x)$ for $x \in [a, b]$. If $f'(a) < k < f'(b)$, then find $g'(c)$.

(p) "Suppose $f : [0, 2] \to \mathbb{R}$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$, with $f(0) = 0$; $f(2) = 1$. If there exists $c \in (0, 2)$, then $f'(c) = \frac{1}{3}$." Is it true *or* false?

(q) Find $\lim_{x \to 0} \frac{x^2 + x}{\sin 2x}$.

(r) "The function $f(x) = 8x^3 - 8x^2 + 1$ has two roots in $[0, 1]$." Write true **or** false.

2. Answer ***any five*** parts : $2 \times 5 = 10$

(a) Use the definition of limit to show that
$$\lim_{x \to 2} \left(x^2 + 4x\right) = 12.$$

(b) Find $\lim_{x \to 0} x \sin\left(\frac{1}{x^2}\right)$, $(x \neq 0)$.

(c) Give an example of a function that has a right-hand limit but not a left-hand limit at a point.

(d) Define $g : \mathbb{R} \to \mathbb{R}$ by
$$g(x) = \begin{cases} 2x & \text{for } x \in Q \\ x + 3; & \text{for } x \in Q^c \end{cases}$$
Find all points at which g is continuous.

(e) Show that the '*sine*' function is continuous on $\mathbb{R}$.

(f) Show that the function $f(x) = \frac{1}{x}$ is uniformly continuous on $[a, \infty]$, where $a > 0$.

(g) Using the mean value theorem, show that
$$\frac{x-1}{x} < \ln(x) < x - 1 \text{ for } x > 1.$$

(h) Show that $f(x) = x^{1/3}$, $x \in \mathbb{R}$, is not differentiable at $x = 0$.

(i) Let $f(x)=\frac{\ln(\sin x)}{\ln(x)}$

Find $\lim_{x\to 0^+} f(x)$.

(j) State Darboux's theorem.

3. Answer ***any four*** parts : 5×4=20

(a) Prove that a number $c\in\mathbb{R}$ is a cluster point of a subset A of $\mathbb{R}$ if and only if there exists a sequence (x_n) in A such that $\lim_{n\to\infty} x_n = c$ and $x_n \neq c$ for all $n\in\mathbb{N}$.

(b) State and prove squeeze theorem.

(c) Let $A\subseteq\mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let f and g be continuous at a point c in A. Prove that $f-g$ and fg are continuous at c.

(d) Give an example of functions f and g that are both discontinuous at a point c in $\mathbb{R}$ such that $f+g$ and fg are continuous at c.

(e) If $f : A\to\mathbb{R}$ is a Lipschitz function, then prove that f is uniformly continuous on A.

(f) Determine where the function

$$f(x)=|x|+|x-1|$$

from $\mathbb{R}$ to $\mathbb{R}$ is differentiable and find the derivative.

(g) Find $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

(h) Determine whether *or* not $x=0$ is a point of relative extremum of the function $f(x)=x^3+2$.

4. Answer ***any four*** parts : 10×4=40

(a) Let $f : A\to\mathbb{R}$ and let c be a cluster point of A. Prove that the following are equivalent :

(i) $\lim_{x\to c} f(x)=L$

(ii) Given any ε-neighbourhood $V_\varepsilon(L)$ of L, there exists a δ-neighbourhood $V_\delta(c)$ of c such that if $x \neq c$ is any point $V_\delta(c) \cap A$, then $f(x)$ belongs to $V_\varepsilon(L)$.

(b) (i) Find $\lim\limits_{x \to 0} \frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2}$, where $x > 0$. 4

(ii) Prove that $\lim\limits_{x \to 0} \cos\left(\frac{1}{x}\right)$ does not exist but $\lim\limits_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0$. 6

(c) (i) Let $f(x) = e^{\frac{1}{x}}$ for $x \neq 0$. Show that $\lim\limits_{x \to 0^+} f(x)$ does not exist in $\mathbb{R}$ but $\lim\limits_{x \to 0^-} f(x) = 0$. 5

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x) + f(y)$ for all x, y in $\mathbb{R}$. Suppose that $\lim\limits_{x \to 0} f(x) = L$ exists. Show that $L = 0$ and then prove that f has a limit at every point c in $\mathbb{R}$. 5

(d) (i) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is not continuous at any point of $\mathbb{R}$. 5

(ii) Prove that every polynomial function is continuous on $\mathbb{R}$. 5

(e) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $|f|$ be defined by $|f|(x) = |f(x)|$ for $x \in A$. Also let $f(x) \geq 0$ for all $x \in A$ and let $\sqrt{f}$ be defined by $\left(\sqrt{f}\right)(x) = \sqrt{f(x)}$ for $x \in A$. Prove that if f is continuous at a point c in A, then $|f|$ and $\sqrt{f}$ are continuous at c. 5+5=10

(f) (i) State and prove Bolzano's intermediate value theorem. 1+4=5

(ii) Let A be a closed bounded interval and let $f : A \to \mathbb{R}$ is continuous on A. Prove that f is uniformly continuous on A. 5

(g) Let $A \subseteq \mathbb{R}$ be an interval, let $c \in A$, and let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions differentiable at c. Prove that

(i) the function $f+g$ is differentiable at c and

$$(f+g)'(c) = f'(c) + g'(c)$$ 5

(ii) if $g(c) \neq 0$, then the function f/g is differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)\,g(c) - f(c)\,g'(c)}{(g(c))^2}$$ 5

(h) State and prove Rolle's theorem. Give the geometrical interpretation of the theorem. (2+5)+3=10

(i) (i) Use Taylor's theorem with $n = 2$ to approximate $\sqrt[3]{1+x},\ x > -1$. 5

(ii) If $f(x) = e^x$, show that the remainder term in Taylor's theorem converges to zero as $n \to \infty$ for each fixed x_0 and x. 5

(j) Find the limits : 5+5=10

(i) $\lim\limits_{x \to 0^+} x^{\sin x}$

(ii) $\lim\limits_{x \to \frac{\pi}{2}^-} \dfrac{\tan x}{\sec x}$

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